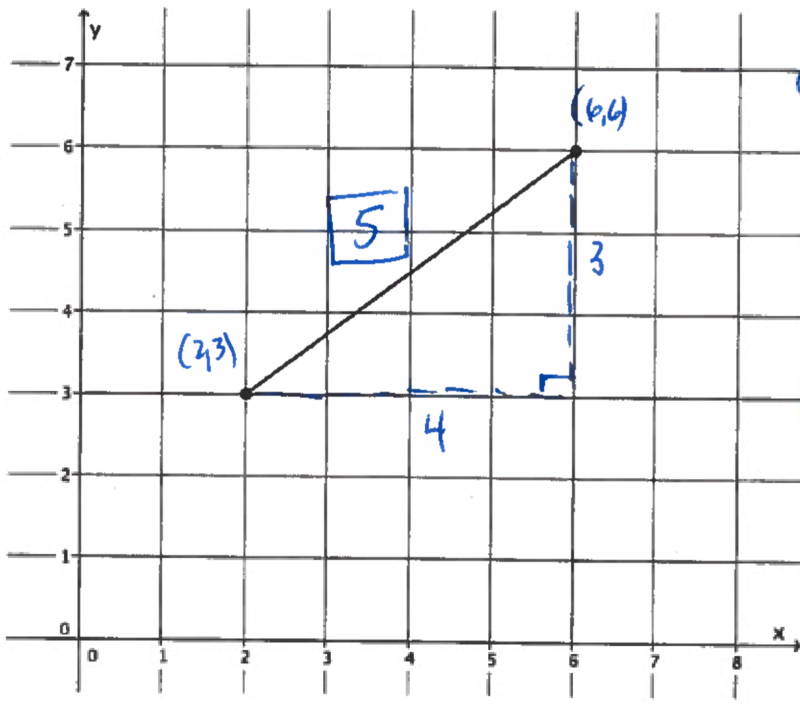
Unit 11
More on Circles – Equations, Arc Lengths and Area Problems

Lesson 1: Writing the Equation of a Circle

Opening Exercise

a. Find the length of the line segment shown on the coordinate plane below.



Two ways:

① Distance Formula

$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

$d = \sqrt{(6-2)^2 + (6-3)^2}$

$d = \sqrt{4^2 + 3^2}$

$d = \sqrt{25}$

$\boxed{d = 5}$

② Make a right Δ/Pyth. Th.

$c^2 = a^2 + b^2$

$c^2 = 3^2 + 4^2$

$c^2 = 25$

$c = \pm\sqrt{25}$

$\boxed{c = 5}$

b. Using the distance formula, find the distance between the points $(9,15)$ and $(3,7)$.

$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

$d = \sqrt{(9-3)^2 + (15-7)^2}$

$d = \sqrt{6^2 + 8^2}$

$d = \sqrt{100}$

$\boxed{d = 10}$

Example 1

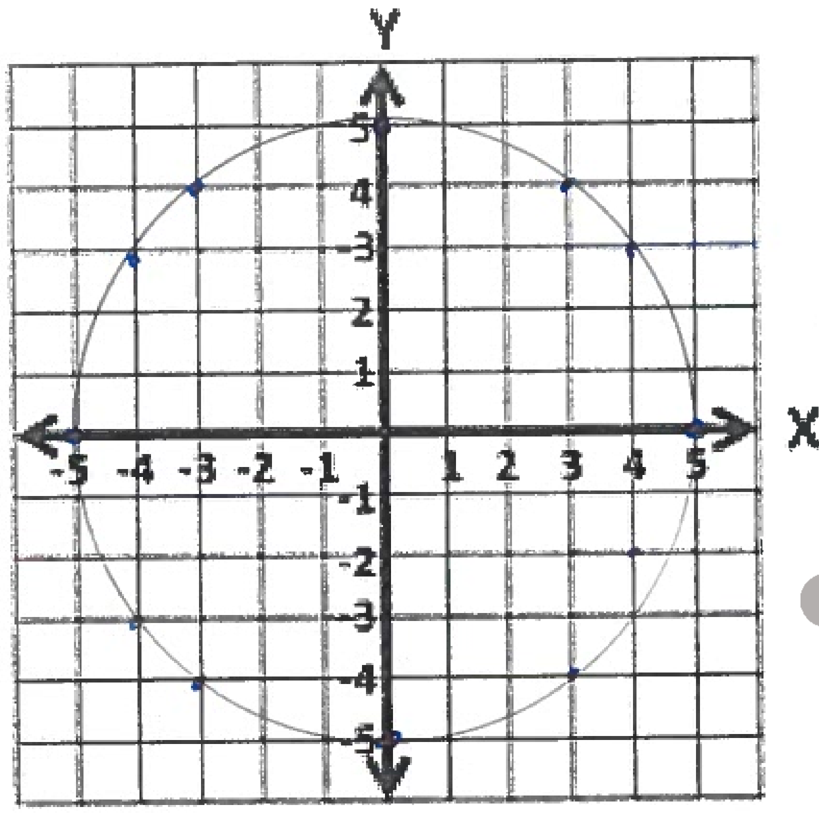
If we graph all of the points whose distance from the origin is equal to 5, what shape will be formed?

Using the given coordinate plane, plot 4 points that are 5 units away from the origin.

Now, we need to find 4 more.

Write down any ideas that you might have to find the location of the next point that is also 5 units from the origin.

(3,4)
(-3,4)
(-3,-4)
(3,-4)



Compare your plan with a partner. Once you agree on a plan, plot three more points using this method. Using your compass, connect these points to form a circle.

In the above circle, the center is located at (0, 0) and the radius length is 5.

We found the location of a point on the circle by using Pythagorean Theorem, which states $a^2+b^2=5^2$.

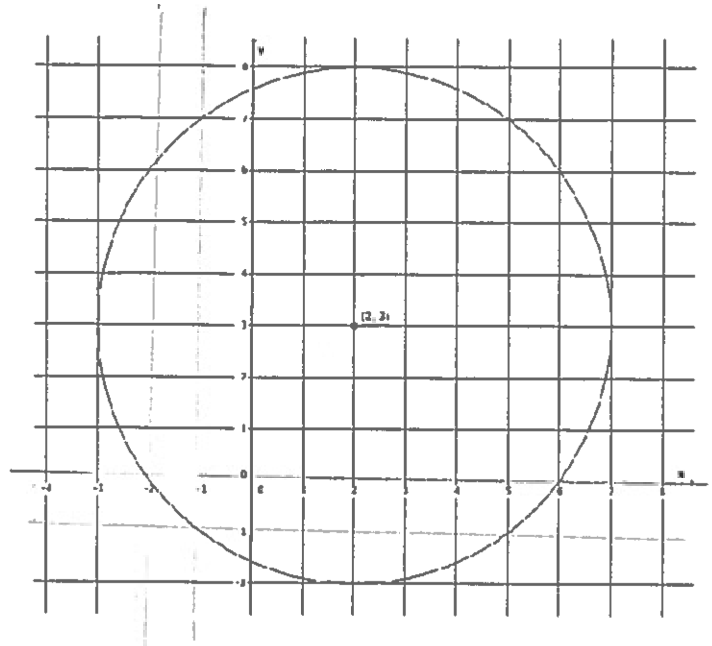
If we generalize this formula by using a point named (x, y), the point will satisfy the equation $x^2+y^2=5^2$ when the circle has a center at the origin.

The equation of a circle whose center is the origin and whose radius is r is

$x^2+y^2=r^2$

Example 2

Now, let's look at a circle that is not centered at the origin.



This circle is centered at $(2,3)$ and has a radius length of 5 units.

Is this circle congruent to the circle we constructed? yes

Is there a sequence of basic rigid motions that would take this circle center to the origin? Explain.

$T_{-2,-3}$

The equation for this circle can be found using this same pattern to move the center of the circle back to the origin.

The equation of this circle is: $(x-2)^2+(y-3)^2=5^2$

☆ note the translation and the number here ☆

Standard Form of the Equation of a Circle

$$(x-a)^2+(y-b)^2=r^2$$ with center (a,b) and radius length r

opposite of center

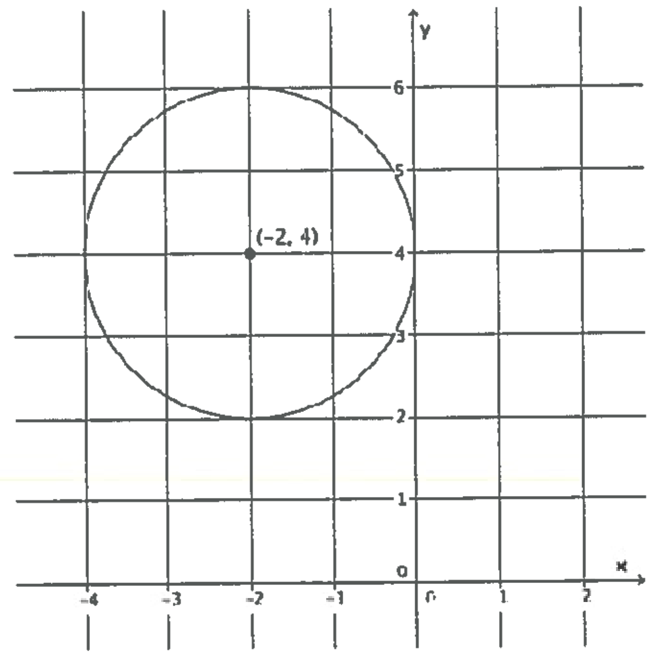
Example 3

Write the equation of the circle that is graphed below.

Think of the translation that moves the center to the origin.

- What do you notice?

opposite of the coordinates $\rightarrow T_{2,-4}$



$(x+2)^2+(y-4)^2=2^2$

$\boxed{(x+2)^2+(y-4)^2=4}$

Example 4

Find the radius and center of the circle given by the equation: $(x+12)^2+(y-4)^2=81$

$C(-12,4)$ ← opposite the values in eq

$r=\sqrt{81}=9$

Example 5

Write an equation for the circle whose center is at $(9,0)$ and has radius 7.

$(x-9)^2+(y-0)^2=7^2$

$\boxed{(x-9)^2+y^2=49}$

Lesson 2: Writing the Equation of a Circle II

Opening Exercise

Two points in the plane, $A(-3,8)$ and $B(17,8)$, represent the endpoints of the diameter of a circle.

a. What is the center of the circle? Explain.

Center is midpoint

$$C = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{-3+17}{2}, \frac{8+8}{2}\right)$$

$$= \left(\frac{14}{2}, \frac{16}{2}\right)$$

$$= (7,8)$$

b. What is the radius of the circle? Explain.

radius is ½ diameter

$$d = \sqrt{(17+3)^2+(8-8)^2}$$

$$d = \sqrt{400} = 20$$

$$r = 10$$

c. Write the equation of the circle.

$$(x-7)^2+(y-8)^2=10^2$$

$$\boxed{(x-7)^2+(y-8)^2=100}$$

Example 1

Write the equation of a circle with center $(3,10)$ that passes through $(12,12)$?

$$r = \sqrt{\Delta x^2 + \Delta y^2}$$

$$r = \sqrt{(12-3)^2 + (12-10)^2}$$

$$r = \sqrt{9^2 + 2^2} = \sqrt{85}$$

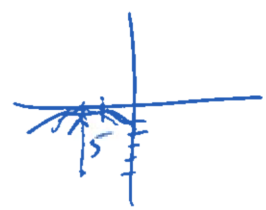
$$(x-3)^2 + (y-10)^2 = (\sqrt{85})^2$$

$$\boxed{(x-3)^2 + (y-10)^2 = 85}$$

Example 2

A circle with center $(2,-5)$ is tangent to the x-axis.

a. What is the radius of the circle? – tangent to x-axis ∴ $r = 5$



b. What is the equation of the circle? $(x-2)^2 + (y+5)^2 = 5^2$

$(x-2)^2 + (y+5)^2 = 25$

Example 3

Given a circle centered at the origin that goes through point (0, 2), determine whether or not this circle would go through the point $(1, \sqrt{3})$.

$(x-a)^2+(y-b)^2=r^2$

$(0-0)^2+(2-0)^2=r^2$

$4=r^2$

$2=r$

$x^2+y^2=4$

$1^2+(\sqrt{3})^2 \stackrel{?}{=} 4$

$1+3 \stackrel{?}{=} 4$

$4=4$

yes, it would go thru point $(1,\sqrt{3})$

Example 4

Determine the center and radius of each circle:

a. $(x+4)^2+(y-6)^2=50$

$\boxed{C(-4,6) \quad r=\sqrt{50}}$

b. $\frac{3x^2}{3}+\frac{3y^2}{3}=\frac{75}{3}$

$x^2+y^2=25$

$\boxed{C(0,0)}$

$r=\sqrt{25}=$

$\boxed{r=5}$

c. $4(x-2)^2+4(y-9)^2-64=0$

$\frac{4(x-2)^2}{4}+\frac{4(y-9)^2}{4}=\frac{64}{4}$

$(x-2)^2+(y-9)^2=16$

$\boxed{C(2,9)}$

$r=\sqrt{16}$

$\boxed{r=4}$

Homework

1. Determine the center and radius of the circle $2(x+1)^2+2(y+2)^2=10$.

2. Write the equation of a circle that has a center of (-4, -3) and is tangent to the *y*-axis.

3. A circle has a diameter with endpoints at $(3,-2)$ and $(3,6)$. Write the equation for this circle.

Lesson 3: Recognizing Equations of Circles

Opening Exercise

Complete the following table:

Polynomial	Factored Form
x^2+2x+1	$(x+1)^2$
x^2+4x+4	$(x+2)^2$
x^2-6x+9	$(x-3)^2$
$x^2+8x+16$	$(x+4)^2$
$x^2-14x+49$	$(x-7)^2$
$x^2-20x+100$	$(x-10)^2$

Equation of a Circle

Standard Form	General Form
$(x-a)^2+(y-b)^2=r^2$	$x^2+y^2+Dx+Ey+F=0$

Example 1

Find the center and the radius of the following:

a. $\underbrace{x^2+4x+4}+\underbrace{y^2-6y+9}=36$

$(x+2)^2+(y-3)^2=36$

$\boxed{\begin{matrix} C(-2,3) \\ r=6 \end{matrix}}$

b. $\underbrace{x^2-10x+25}+\underbrace{y^2+14y+49}=4$

$(x-5)^2+(y+7)^2=4$

$\boxed{\begin{matrix} C(5,-7) \\ r=2 \end{matrix}}$

Example 2

Find the center and the radius of the following: $x^2+4x+y^2-12y=41$

Use completing the square $\rightarrow$ $x^2+4x+\underline{4}+y^2-12y+\underline{36}=41+\underline{4}+\underline{36}$

$\tfrac{1}{2} \to (+2)^2$ $\quad \tfrac{1}{2} \to (-6)^2$

$(x+2)^2+(y-6)^2=81$

$\boxed{\begin{matrix} C(-2,6) \\ r=9 \end{matrix}}$

$x^2+bx=c$

$x^2+bx+\underline{\tfrac{b^2}{4}}=c+\underline{\tfrac{b^2}{4}}$

① take ½ → $\left(\pm\tfrac{b}{2}\right)^2$ ② square it.

use the sqr always!

$\left(x\pm\tfrac{b}{2}\right)^2=$ ____

Example 3

Could the circle with the equation $x^2-6x+y^2-7=0$ have a radius of 4? Why or why not?

$x^2-6x+\underline{9}+y^2=7+\underline{9}$

$(-3)^2$

$(x-3)^2+y^2=16$

yes

Example 4

Identify the graphs of the following equations as a circle, point, or an empty set.

a. $x^2+y^2+4x=0$

$x^2+y^2+4x=0$

$x^2+4x+\underline{4}+y^2=0+\underline{4}$

$+2$

$(x+2)^2+y^2=4$

circle

$\boxed{C(-2,0) \quad r=2}$

b. $x^2+y^2+6x-4y+15=0$

$x^2+6x+\underline{9}+y^2-4y+\underline{4}=-15+\underline{9}+\underline{4}$

$+3$ -2

$(x+3)^2+(y-2)^2=-2$

No!

empty set

① Complete square

② Constant on right.

1) Positive - circle

2) 0 → pt.

3) Negative - empty

Summary

When r^2 is ...	The figure is ...
Positive	circle
Negative	empty set
Zero	point

Exercises

1. The graph of the equation below is a circle. Identify the center and radius of the circle.

$$x^2+10x+y^2-8y-8=0$$

$x^2+10x+\underline{25}+y^2-8y+\underline{16}=8+\underline{25}+\underline{16}$

$+5^2$ $\quad -4^2$

$(x+5)^2+(y-4)^2=49$

$\boxed{c(-5,4) \quad r=7}$

2. Identify the graphs of the following equations as a circle, point, or an empty set.

   a. $x^2+2x+y^2=-1$

   $x^2+2x+\underline{1}+y^2=-1+\underline{1}$

   $+1^2$

   $(x+1)^2+y^2=0$ pt.

   b. $x^2+y^2=-3$ empty set

   c. $x^2+y^2+6x+6y=7$

   $x^2+6x+\underline{9}+y^2+6y+\underline{9}=7+\underline{9}+\underline{9}$

   $+3^2$ $\quad +3^2$

   $(x+3)^2+(y+3)^2=25$ Circle

Example 5

Chante claims that two circles given by $(x+2)^2+(y-4)^2=49$ and $(x-3)^2+(y+8)^2=36$ are externally tangent. She is right. Show that she is.

$(x+2)^2+(y-4)^2=49$ $\quad$ $(x-3)^2+(y+8)^2=36$ $\quad$ $(x+2)^2+(y-4)^2=(x^2-3)^2+(y+8)^2+13$

$c(-2,4)$ $\quad$ $c(3,-8)$

$r_1=7$ $\quad$ $r_2=6$

the distance of centers:

$d=\sqrt{\Delta x^2+\Delta y^2}$

$r_1+r_2=13$ $\quad$ $d=\sqrt{(3+2)^2+(-8-4)^2}$

$d=\sqrt{25+144}$

same $\quad$ $d=\sqrt{169}=13$

Homework

1. Identify the center and radius of the following circles.

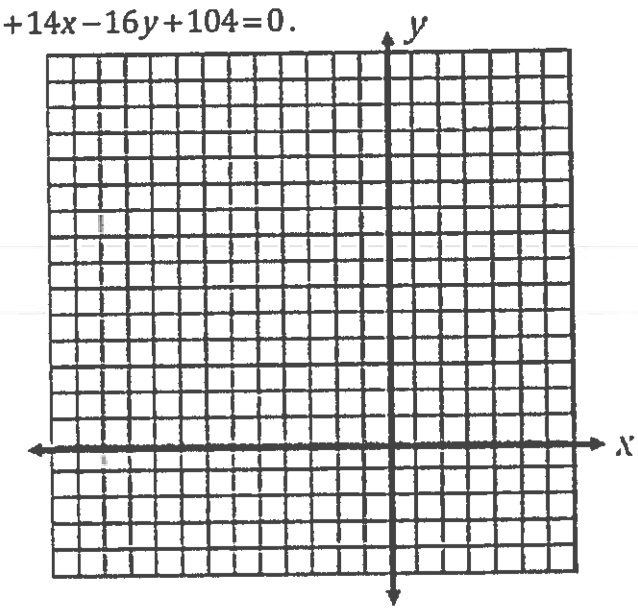
   a. $(x-25)^2+y^2=1$

   b. $x^2+2x+y^2-8y=8$

   c. $x^2-20x+y^2-10y+25=0$

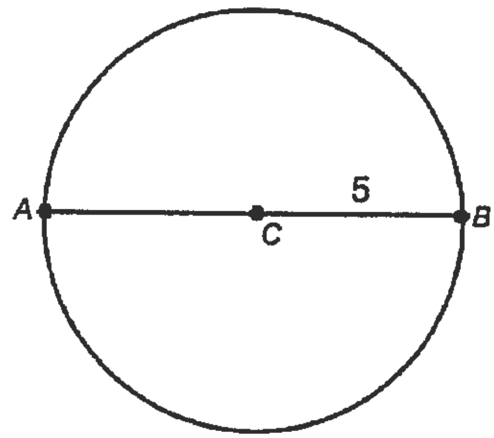
   d. $x^2+y^2=19$

2. Sketch a graph of the equation $x^2+y^2+14x-16y+104=0$.



Lesson 4: Arc Length and Areas of Sectors

Opening Exercise



a. How many degrees make up a full rotation of a circle? $360°$

b. How many degrees are in the measure of $\widehat{AB}$? $180°$

c. What is the measure of $\angle ACB$? $180°$

d. What kind of angle is $\angle ACB$? straight

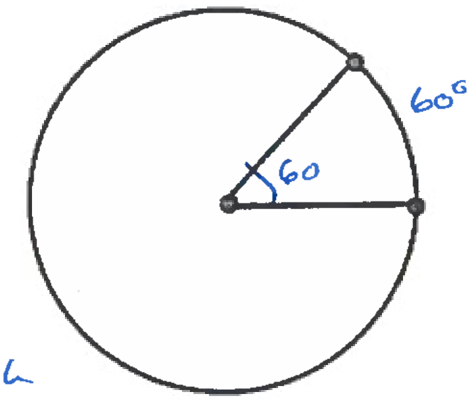
e. Find the exact value of the circumference of the circle. $C = 2\pi r = 10\pi$

f. What is the exact measure of the length of $\widehat{AB}$? $\frac{1}{2}(10\pi) = \underline{5\pi}$

Definition	Diagram
Arc Length • the circular distance around the arc – not measured in degree	

Example 1

Find the exact length of the arc of degree measure $60°$ in a circle of radius 10 cm.



The arc length is part of the circumference

$C = 2\pi r$

$C = 2\pi(10)$

$C = 20\pi$

Measure of the arc is = to the center angle

$\frac{60°}{360°}$ would be the part of the circle for the arc

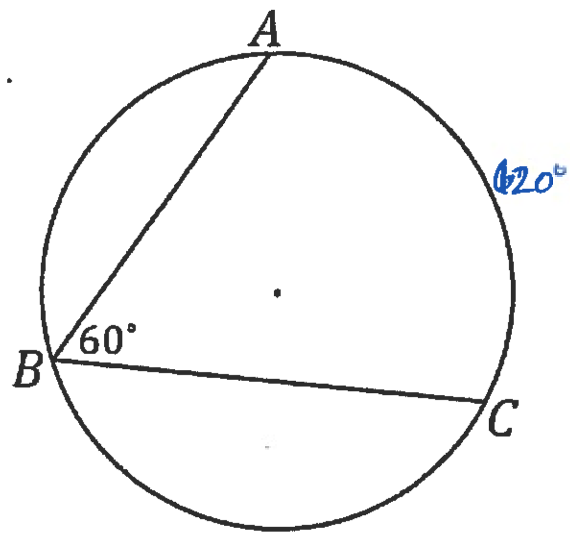
$\frac{1}{6} \times \frac{20\pi}{1} = \frac{20\pi}{6} = \frac{10\pi}{3}$ cm ← arc length

Formula for Arc Length
$L = \frac{n}{360}\pi d$ or $L = \frac{n}{180}\pi r$ ↑ diameter ↖ radius

n - measure of central angle (the degree measure of the arc)

Example 2

The radius of the pictured circle is 36 cm, and $m\angle ABC = 60°$. What is the exact arc length of $\widehat{AC}$?



$L = \frac{n}{360} \pi d$ $\quad$ 2×36

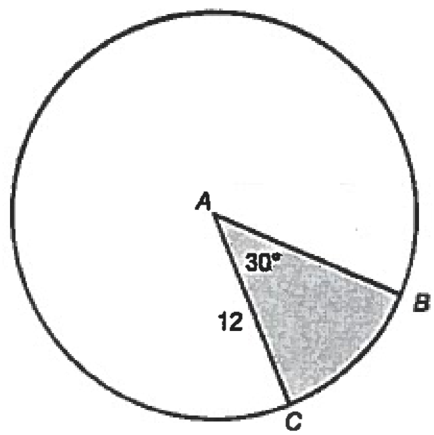
$= \frac{120}{360} \pi (72)$

$= \frac{1}{3} \pi \cdot 72$

$= \boxed{24\pi}$

Example 3

a. Find the length of arc $\widehat{BC}$.



$L = \frac{n}{360} \pi (2 \times 12)$

$= \frac{30}{360} \pi \cdot 24$

$= \frac{1}{12} \pi \cdot 24$

$= \boxed{2\pi}$

b. Using the same concept we used to find arc length, how can we find the area of the shaded region?

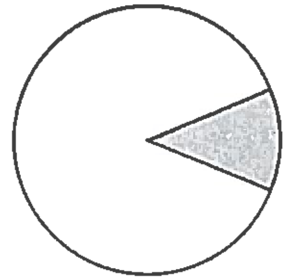
$A = \frac{n}{360} \pi r^2$ $\qquad$ $A = \frac{n}{360}$ Area○

$= \frac{30}{360} \pi (12)^2$

$= \frac{1}{12} \pi \cdot 144 = \boxed{12\pi}$

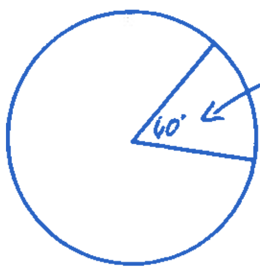
Formula for Area of a Sector

$$A = \frac{n}{360} \pi r^2$$



Example 4

Circle O has a minor arc $\overset{\frown}{AB}$ with an angle measure of $60°$. Sector AOB has an area of 24π. What is the length of the radius of circle O?



$A = 24\pi$

$$A = \frac{n}{360} \pi r^2$$

$$24\cancel{\pi} = \frac{60}{360} \cancel{\pi} r^2$$

$$6 \times \left[24 = \frac{1}{6} r^2 \right] \times 6$$

$$144 = r^2$$

$$r = \pm\sqrt{144}$$

$$\boxed{r = 12}$$

Exercises

1. The area of sector AOB in the following diagram is 28π and the radius is 12 cm. Find the measure of $\angle AOB$.

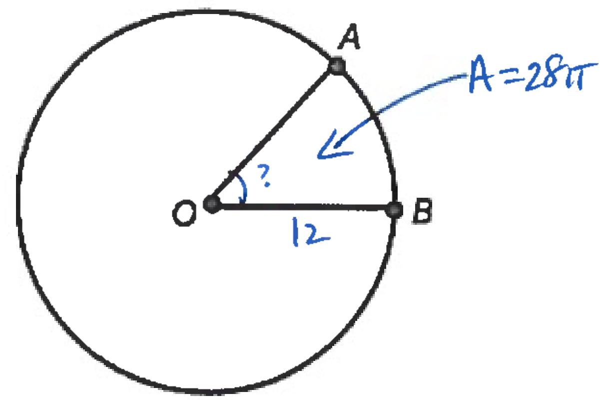
$A = \frac{n}{360}\pi r^2$

$28\cancel{\pi} = \frac{n}{360}\cancel{\pi}(12)^2$

$360 \times \left[28 = \frac{144n}{360} \right]$

$\frac{10080}{144} = \frac{144n}{144}$

$\boxed{n = 70^\circ}$



2. In the following figure, circle O has a radius of 8 cm, $m\angle AOC = 108^\circ$, and $AB = AC = 10\text{cm}$. Find:

a. $m\angle OAB$

$\triangle AOC \cong \triangle AOB$ (SSS)

$\overset{\frown}{BA} \cong \overset{\frown}{AC}$ ($\cong$ chords $\rightarrow \cong$ arcs)

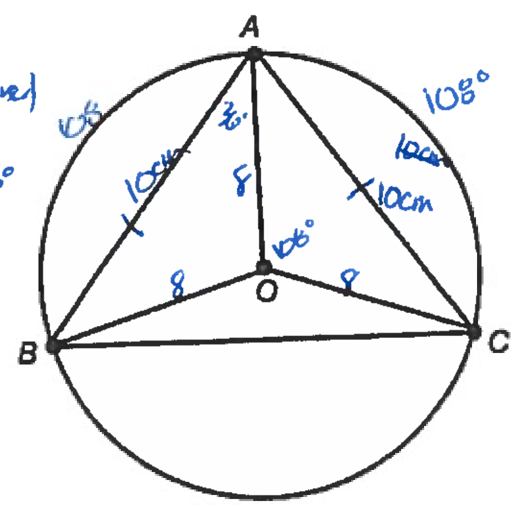
$\therefore m\angle AOB$ is 108°

$\triangle AOB$ is isos. $\therefore$ $180 - 108 = 72 \div 2 = 36^\circ$

b. $m\overset{\frown}{BC}$

$108 \times 2 = 216$

$360 - 216 = 144^\circ$



c. Area of sector BOC.

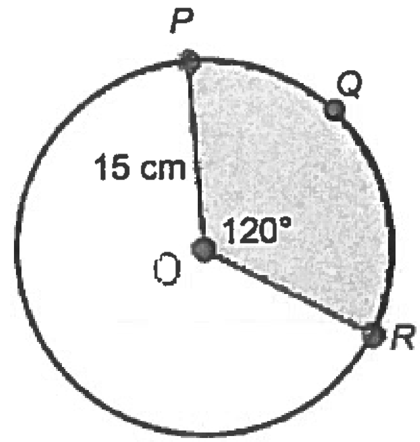
$A = \frac{n}{360}\pi r^2$

$A = \frac{144}{360}\pi(8)^2$

$A = 80.4247719$

Homework

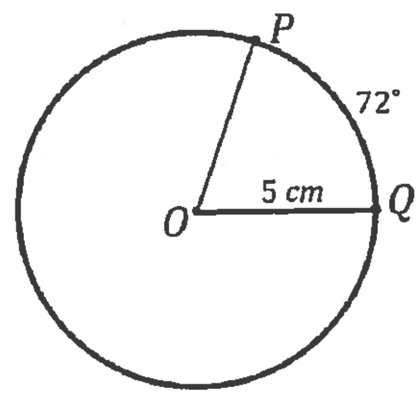
1. *a.* Find the exact value of the arc length of $\widehat{PQR}$.



b. Find the exact area of sector POR.

2. Using the picture of circle O shown, determine the following to the *nearest tenth*:

a. arc length of $\widehat{PQ}$



b. area of sector POQ

Lesson 5: Unknown Length and Area Problems

Opening Exercise

1. Find the exact area of the shaded region.

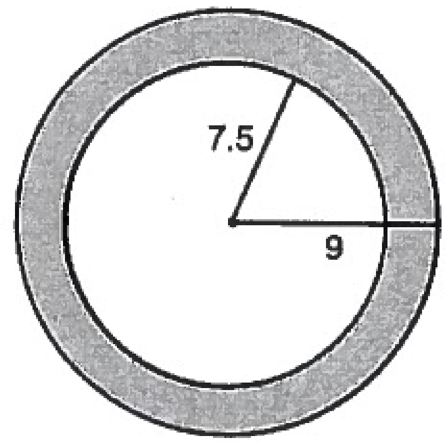
$A_{shade} = A_{big\,O} - A_{small\,O}$

πr^2

$= \pi(9)^2 - \pi(7.5)^2$

$= 81\pi - 56.25\pi$

$= \boxed{24.75\pi}$



2. Find the area of the shaded region to the *nearest tenth*.

$A_{shade} = A_{sector} - A_{\triangle}$

$= \frac{n}{360}\pi r^2 - \frac{1}{2}bh$

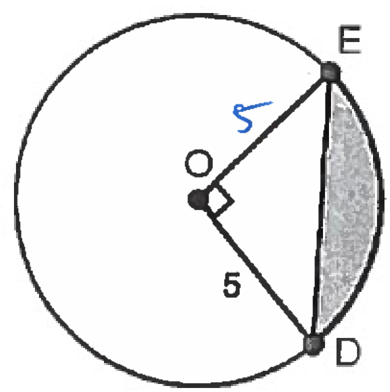
$= \frac{90}{360}\pi(5)^2 - \frac{1}{2}(5)(5)$

$= \frac{1}{4}\pi \cdot 25 - \frac{1}{2}(25)$

$= \boxed{6.25\pi - 12.5}$

$= 7.134954085$

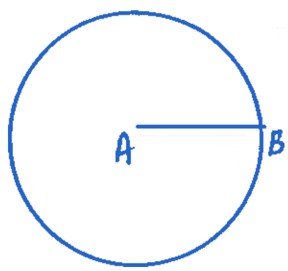
$= \boxed{7.1}$



Example 1

Another way to measure angles:

1. Draw circle A of any size.
2. Draw in radius AB.
3. Measure the radius using the string provided to you.
4. Using your string as a measuring tool, measure and mark the number of strings needed to go around the circle once.



$\frac{2\pi r}{r}$

Approximately how many strings did it take to make it all the way around the circle? Was this the same for everyone in the class?

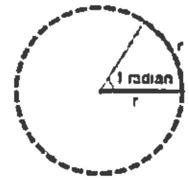
6.28

What is the relationship between the circumference and radius?

$C = 2\pi r$

What does this really mean?

The central angle that intercepts two consecutive string markings on your arc is equal to 1 radian.

Definition	Diagram
Radian • the measure of a central angle when the arc it subtends is equal in length to the radius	

In Example 1, we saw that it takes 2π radii to go all the way around any circle. ($C = 2\pi r$)

Therefore, 2π radians = $360°$.

How can we determine the number of degrees there are in 1 radian?

$2\pi r = 360$

$1r = \frac{360}{2\pi}$

$1r = \frac{180}{\pi}° \approx 57.29578°$

Formulas (on Reference Sheet!)

Radians	**Degrees**
$1 \text{ radian} = \frac{180}{\pi} \text{ degrees}$	$1 \text{ degree} = \frac{\pi}{180} \text{ radians}$

To convert degrees → radian: $\times \frac{\pi}{180}$

To convert radians → degrees: $\times \frac{180}{\pi}$

Example 2

Circle B has a radius of 10 cm. and the measure of central angle B is 1.5 radians. Find the length of the intercepted arc.

$L = \frac{n}{360} \pi d$

$L = \frac{n}{360} \cdot \pi \cdot 2r$

$L = \frac{n}{360} \cdot 2\pi \cdot r$

$360° = 2\pi$ $\quad L = \frac{n}{2\pi} \cdot 2\pi r$

$L = n r$

radian measure

radius

this is normally written

$S = r\theta$

arc length

radian central ∡

$S = r\theta$

$S = 10(1.5)$

$\boxed{S = 15\text{cm}}$

Lesson 6: Unknown Length and Area Problems

Opening Exercise

Circle B has a radius of 14 cm. Angle B intercepts the arc with a length of 6π. Find the measure of angle B in radians.

$$S = r\theta$$

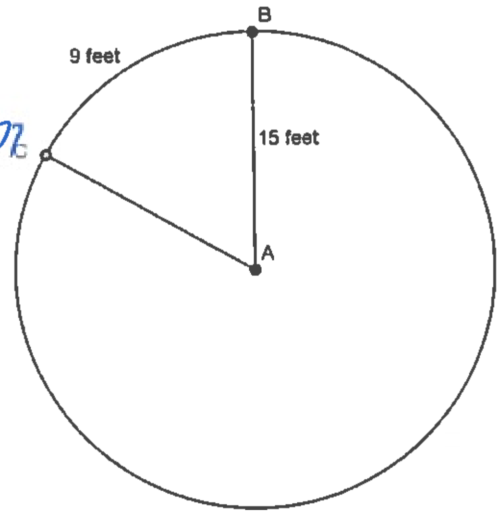
$$\frac{6\pi}{14} = \frac{14\theta}{14}$$

$$\theta = \frac{6}{14}\pi$$

$$\boxed{\theta = \frac{3}{7}\pi}$$

Exercises

1. Given circle A, find to the *nearest hundredth*:



a. $m\overset{\frown}{BC}$ in degrees.

$S = r\theta$

$9 = 15\theta$

$\frac{9}{15} = \theta$

$\frac{9}{15} \cdot \frac{180}{\pi} = 34.377$

$\boxed{34.38^\circ}$

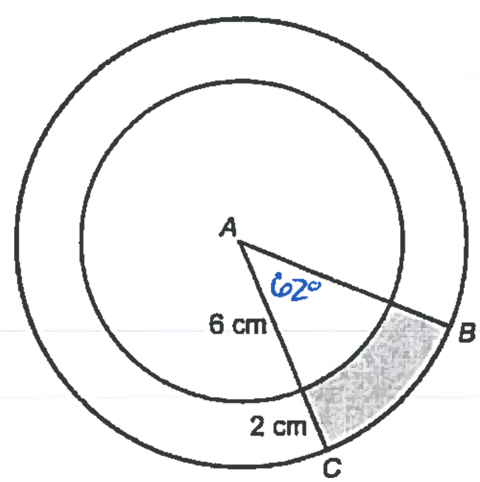
b. the area of sector BAC.

$A = \frac{n}{360}\pi r^2$

$A = \frac{34.38}{360}\pi(15)^2$

$\boxed{A = 67.5}$

2. Find the area of the shaded region to the *nearest hundredth* if the $m\angle BAC = 62^\circ$.



$A_{shade} = A_{big\ sect} - A_{small\ sect}$

$= \frac{62}{360}\pi(8)^2 - \frac{62}{360}\pi(6)^2$

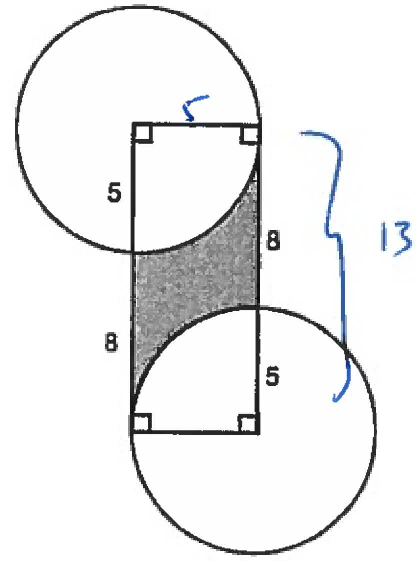
$= 15.1494571$

$\approx \boxed{15.15\ cm^2}$

$\frac{62}{360}\pi\left[8^2 - 6^2\right]$

$\boxed{\frac{n}{360}\pi\left[R^2 - r^2\right]}$

3. Find the area of the shaded region to the *nearest hundredth*.



$A_{shade} = A_{\square} - A_{\triangle} - A_{\triangle}$

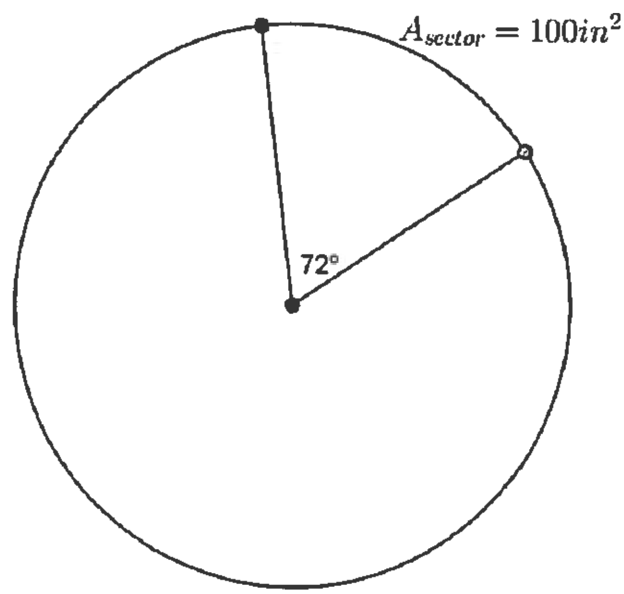
$= 13(5) - \frac{1}{4}\pi(5)^2 - \frac{1}{4}\pi(5)^2$

$= 65 - \frac{25}{4}\pi - \frac{25}{4}\pi$

$= 25.73009183$

$\approx \boxed{25.73}$

4. Find the area of the entire circle given the area of the sector.



$A = \frac{n}{360}\pi r^2$

$\left[\ 100 = \frac{72}{360}\underbrace{\pi r^2}_{\text{area of circle}}\ \right] \times \frac{360}{72}$

$\boxed{500} = A = \pi r^2$